2 Basic properties of autonomous first order ODE

I start with some mathematical properties of the first order ODE and after this will show how one can put them to a good use analyzing an ecological problem devoted to optimal strategies of harvesting.

2.1 Definitions and basic properties

Definition 1. *First order ODE $\dot{x} = f(t, x)$ is called autonomous if the right hand side does not depend explicitly on t:*

$$\dot{x} = f(x). \tag{1}$$

I start with an example, and then generalize the properties deduced in this example to all autonomous equations.

Example 2. Consider the simplest autonomous equation

$$\dot{x} = x,$$

which is a separable equation, and whose solution is

$$x(t) = Ce^t,$$

where C is an arbitrary constant. If I had the initial condition $x(t_0) = x_0$, then my solution would be

$$x(t; x_0) = x_0 e^{t-t_0}.$$

If one has the explicit formula for the solution then it is easy to sketch several integral curves (i.e., the graphs of the solutions) and the direction field of this equation (see the figure).

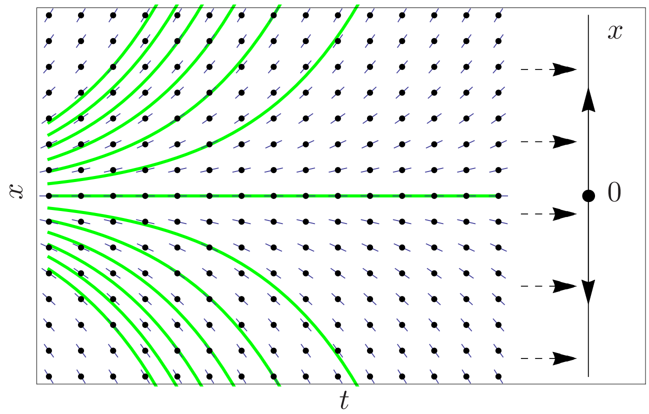

Figure 1: The direction field of $\dot{x} = x$ together with the phase line. The dashed arrows show the projection of the direction field onto the x-axis.

From the figure it becomes obvious that the direction field of the equation in the example, as well as any direction field of the autonomous equations, have the property that it is the same on any line

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parallel to t axis (recall that the right hand side does not depend on t). Hence I can project the whole direction field onto the x-axis, without losing much information (I put an arrow that points in the positive direction if the slope is positive and an arrow that points in the negative direction if the slope is negative, what lost is the absolute values of the slopes). The picture on the x-axis is called the *phase portrait* and the x axis is called the *phase line* (again, the terminology originated in mechanics). Note that for $x = 0$ there is no direction and hence I mark this point on the phase line.

Can I come up with the phase portrait without looking at the direction field, which was known to me because of the simplicity of the original equation? The answer is resounding "yes." Consider the following Fig. 2. Here I look at the graph of function f, which is simply $f(x) = x$ in my case. Note that if $x > 0$ then $f(x) > 0$ hence $\dot{x} > 0$ hence the solutions are increasing. Similarly, if $x < 0$, then $\dot{x} < 0$ and solutions are decreasing, I point these facts with arrows in the graph and obtain again the same phase portrait that I already saw in the previous figure. Hence the conclusion: We do not need actual analytical (i.e., in the form of a formula) solutions to the autonomous differential equations to figure out the phase portrait of this equation. And knowledge of the phase portrait allows me to infer the asymptotic behavior of the solutions ("asymptotic" in this context means for $t \to \infty$). In my example I see that if the initial condition $x_0 > 0$ then $x(t; x_0) \to \infty$ for $t \to \infty$, and if $x_0 < 0$ then $x(t; x_0) \to -\infty$. Here and throughout the course the notation $x(t; x_0)$ means the solution to ODE with the initial condition x_0.

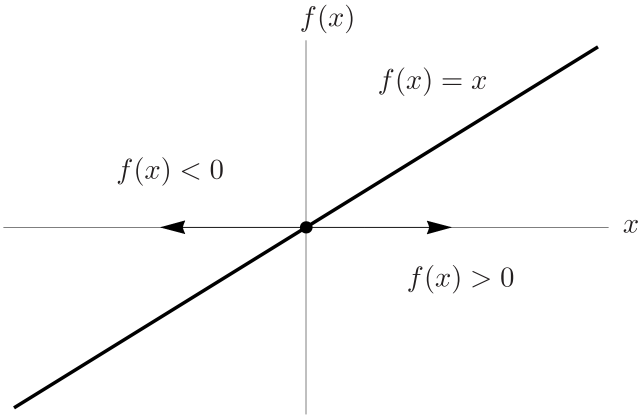

Figure 2: The phase portrait of $\dot{x} = x$.

Problem 1. Analyze, by building the phase portrait, asymptotic behavior of the solutions to $\dot{x} = -x$.

Example 3. Here is another example. Recall that the logistic equation has the form

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \quad r, K > 0.$$

Here r, K are positive parameters. I actually wrote its solutions in the previous lecture. Here I will sketch several integral curves by studying its phase portrait. The graph of $f(x) = rx(1 - x/K)$ is a parabola with branches pointing down, which crosses x-axis at the points $\hat{x}_1 = 0$ and $\hat{x}_2 = K$ (see Fig. 3, left panel). I can see that $f(x)$ is negative when $x < 0$ and $x > K$ and positive for $x \in (0, K)$, hence the directions of the arrows. This means that the integral curves are growing for $x \in (0, K)$ and degreasing for $x < 0$ and $x > K$. If $x = 0$ or $x = K$ I have that $f(x) = 0$, and hence the slope of the integral curves here is zero, these points in the phase correspond to the integral curves parallel t-axis. Having just this information I can sketch several integral curves (see Fig. 3, right panel, you should compare it with the left panel and clearly understand how they connected).

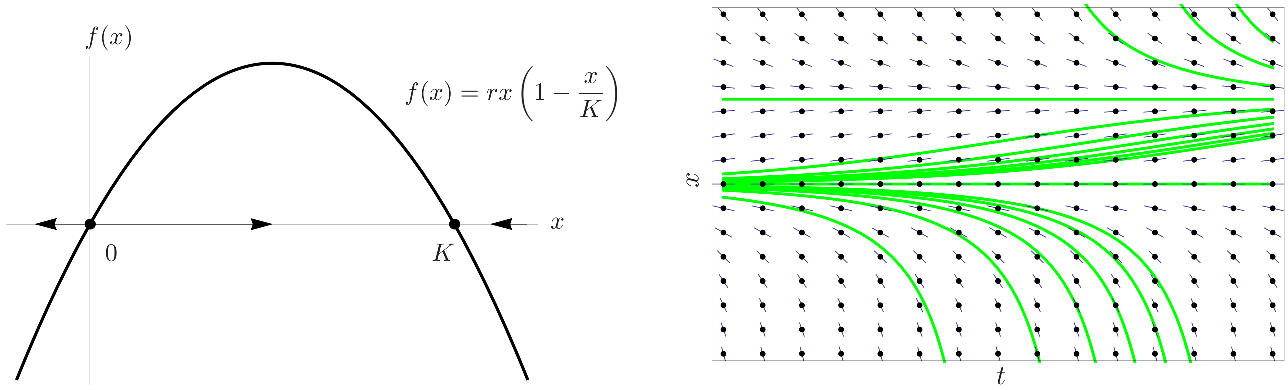

Figure 3: Left: The phase portrait of $\dot{x} = rx(1-x/K)$. Right: The direction field of $\dot{x} = rx(1-x/K)$. The two horizontal integral curves are $x = 0$ and $x = K$.

Now I am in the position to formulate some basic general properties of the autonomous ODE.

- The direction field is invariant with respect to translations along t-axis. This is the reason I can still get a lot of information simply from the phase portrait (i.e., from the projection of solution curves on the phase line).

- A related property is that if $t \mapsto x(t)$ solves the problem (1) then $t \mapsto x(t + c)$, where c is any constant, also a solution. This means that if I know the solution with the initial condition $x(0) = x_0$, then any solution with the initial condition $x(t_0) = x_0$ can be obtained by translation. This is why I can write $x(t; x_0)$ without specifying the time moment at which the initial condition is prescribed.

- The solutions to the autonomous equation are monotonous functions. In particular, the first order autonomous equations cannot have periodic solutions. (Can you see why?)

- There are special and very important solutions, which can be found as the roots of $f(x) = 0$. If $\hat{x}$ is such that $f(\hat{x}) = 0$ then $\hat{x}$ is called an *equilibrium point* (or stationary point, or critical point, or rest point, or simply equilibrium, or fixed point). If $\hat{x}$ is an equilibrium, then $x(t) = \hat{x}$ is a solution to (1), which corresponds to the integral curve parallel to the t-axis (look at the examples above).

- The asymptotic behavior of the solutions to the autonomous ODE (1) can be inferred from the phase portrait; there are only three options: Firstly, the solutions can approach the equilibria, secondly, the solutions can be equilibria themselves, and finally, the solutions can go to plus or minus infinity.

- The last point can be rephrased in the following form: the phase portrait is a union of equilibrium points and *orbits* (intervals of $\mathbf{R}$) with specific directions. The orbits are also often called *trajectories*.

If I look again at the example with the logistic equation, I can see that there are two equilibria: $\hat{x}_1 = 0$ and $\hat{x}_2 = K$, but the behavior of the orbits around these points is manifestly different: the point $\hat{x}_1$ repels orbits, whereas $\hat{x}_2$ attracts orbits (look at the directions of the arrows). The mathematical formalization that distinguishes these points is the notion of *stability*. Since this notion is so important for this course, I will give a formal definition.

Definition 4. *An equilibrium $\hat{x}$ of the autonomous first order ODE* (1) *is Lyapunov stable (or simply stable) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for any initial condition x_0 satisfying*

$$|x_0 - \hat{x}| < \delta,$$

it follows that

$$|x(t; x_0) - \hat{x}| < \varepsilon.$$

If, additionally,

$$|x(t; x_0) - \hat{x}| \to 0, \quad t \to \infty,$$

then $\hat{x}$ is called asymptotically stable.

Otherwise, $\hat{x}$ is called unstable.

Asymptotically stable points are called sinks. Unstable points, for which for any x_0 from a neighborhood of $\hat{x}$, the solution $x(t; x_0)$ moves away from $\hat{x}$, are called sources. Also, sinks are called attractors, and sources are called repellers.

This definition is quite general. For the first order equations I mostly will meet with sinks and sources (can you think of an example of a Lyapunov stable equilibrium?). This definition is difficult to apply for concrete examples, since it involves actual solutions to (1). However, even perfunctory inspection of the phase portrait of the logistic equation should bring you the idea of a simple analytical test for stability. The proof is left as an exercise for those who would like to practice their proof writing skills.

Proposition 5. *Let $\hat{x}$ be an equilibrium of* (1)*. Assume that $f \in \mathcal{C}^{(1)}$. If $f'(\hat{x}) > 0$ then $\hat{x}$ is a source; if $f'(\hat{x}) < 0$ then $\hat{x}$ is asymptotically stable or a sink.*

In this proposition $f'(\hat{x})$ means the derivative of f evaluated at the point $\hat{x}$.

For example, if I consider again the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right),$$

then I have

$$f'(x) = r\left(1 - \frac{x}{K}\right) - \frac{rx}{K}.$$

Therefore

$$f'(\hat{x}_1) = f'(0) = r > 0,$$

hence the origin is a source, and

$$f'(\hat{x}_2) = f'(K) = -r < 0,$$

therefore $\hat{x}_2 = K$ is asymptotically stable or a sink.

A very good question to think about is what happens if $f'(\hat{x}) = 0$. Note that this case is not covered by the proposition above.

Definition 6. *An equilibrium $\hat{x}$ of* (1) *such that $f'(\hat{x}) \neq 0$ is called hyperbolic.*

Therefore, if $\hat{x}$ is hyperbolic then I know that it is either sink or source. The converse, however, is not true (can you think of a non-hyperbolic sink?).

2.2 Mathematical models of harvesting

Let me assume that dynamics of a fish population in a lake is governed by the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \quad r, K > 0,$$

such that in the long run I have that the population size stabilizes at $N(t) = K$, the carrying capacity of the lake. Now I assume that I would like to start harvesting the fish in the lake. I need to optimize two conditions: First, we would like to guarantee that the fish does not go extinct in the lake, and second, we would like to maximize the yield. For this there are two strategies:

- *Fixed yield.* This means that I fix the quote for the time period (say, 500 pounds per year).
- *Proportional yield.* This means that I fix the proportion of the fish population I would like to harvest during the time period (say, 25 percent of the current population per year).

A biologically important question is which strategy is better?

Let me start with the fixed yield strategy. The equation that governs the dynamics of population now reads

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - Y_0, \quad r, K, Y_0 > 0,$$

where Y_0 is the yield that I plan to acquire during the time unit. I can find the equation for the equilibrium points and study their stability analytically, but the geometric picture in this case is much easier to deal with. The right hand side here is

$$f(N) = rN\left(1 - \frac{N}{K}\right) - Y_0,$$

whose graph is the parabola defined by $rN\left(1 - \frac{N}{K}\right)$ and shifted down by Y_0. If Y_0 is small enough, then I still have two equilibria, let me call them $\hat{N}_1 > 0$ and $\hat{N}_2 < K$, at that the former is unstable and the latter one is asymptotically stable. My task is to determine the maximum possible Y_0 in terms of the population parameters r, K. It is clear that for some Y_0 the parabola will touch N-axis and after that, for any Y_0 bigger than that, there will be no positive equilibria in the system, which corresponds to extinction (see Fig. 4). Hence, the maximal possible yield corresponds to the moment when parabola exactly touches N-axis, which happens when the discriminant of the quadratic polynomial

$$rN\left(1 - \frac{N}{K}\right) - Y_0 = -\frac{rN^2}{K} + rN - Y_0$$

is equal to zero:

$$r^2 - 4\frac{rY_0}{K} = 0 \implies Y_0 = \frac{rK}{4}.$$

This is the maximal possible yield in this model.

Now back to the proportional yield. The equation reads

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - hN = N\left(r - h - \frac{rN}{K}\right).$$

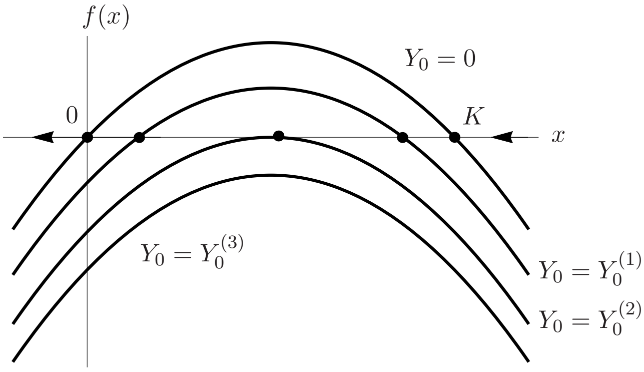

Figure 4: The phase portraits for the model with the fixed yield. $Y_0^{(1)} < Y_0^{(2)} < Y_0^{(3)}$. The maximal possible yield corresponds to $Y_0^{(2)}$, note that in this case we have only one equilibrium. For $Y_0^{(3)}$ there are no positive equilibria and the population goes extinct.

I have here two equilibria:

$$\hat{N}_1 = 0,$$

which is always unstable, and

$$\hat{N}_2 = \frac{K(r-h)}{r},$$

which is always asymptotically stable (check!). To guarantee that $\hat{N}_2 > 0$, I must require that $h < r$. If the population at the stationary point $\hat{N}_2$, then my yield is

$$h\hat{N}_2 = \frac{Kh(r-h)}{r},$$

and I am free to pick any h. Let me maximize the last expression with respect to h: the maximum is attained when $h = r/2$ (prove this using Calculus), hence the maximal yield for this strategy is

$$\frac{rK}{4},$$

which is exactly the same as in the model with fixed yield.

So is there any difference between the two approaches? The answer is "yes, there is." The fixed yield at the maximal value leads to an inevitable catastrophe because sooner or later Y_0 will be such that there will no equilibria. The population will collapse. The mathematical term for this catastrophic event here (two equilibria collide and disappear) is *bifurcation*. For the model with the proportional yield there is no bifurcation (unless I pick $h > r$) since if I somehow exceed the best possible value $h = r/2$, then nothing dramatic happens, I will harvest slightly less fish (see Fig. 5).

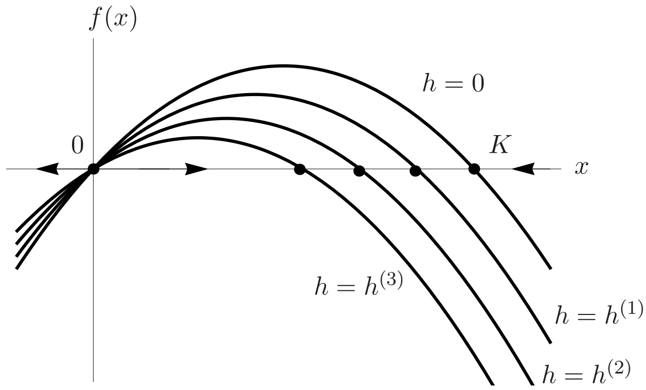

Figure 5: The phase portraits for the model with the proportional yield. $h^{(1)} < h^{(2)} < h^{(3)} < r$. Note that for any $h < r$ we still have two equilibria, the right of which is asymptotically stable.